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Portmanteau Test Statistics in Time Series

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Abstract

This paper reviews some portmanteau test statistics used in time series analysis. It is the first of a series of new statistics to be included in Tol.

Keywords: diagnosis, time series, portmanteau test, serial correlation, nonlinearity.

1. Introduction

One of the most important stages of building a model is that of its diagnosis. In particular, we are interested in finding whether the residuals of our model are white noise, that is, follow process that shows no serial correlation, is homocedastic, etc.

It is obvious that we must have a look at the ACF and PACF of the residuals, apart form the graph of the residuals itself. The graph will show which lags of the ACF and PACF display significative values and will also show some remaining structure that may happen. However, these graphs will show linear dependent structures, and it is well known that in many cases we are likely to have different nonlinear structures.

The second option is to build a test statistic to test the null hypothesis that the residuals are independent up to a lag m. Notice that we are going beyond the usual hypothesis that the first m autocorrelations are zero. In particular, we are interested in finding not only linear, but also nonlinear structures. These tests should be applied in order to check for nonlinearity in mean and also for nonlinearity in variance.

The usual test statistics used in this second approach are the so called portmanteau statistics. The initial portmanteau tests statistics proved to be very inefficient to detect departures from assumptions, but there has been an increasing interest on the field and a new breed of statistics has recently appeared (see Li and McLeod 1981; McLeod 1994; Monti 1994; Hong 1996; Peña and Rodríguez 2002; Peña and Rodríguez 2004; Chen and Deo 2004; Duchesne and Roy 2004; Bouhaddioui and Roy 2004). There are many papers analyzing the behavior of the tests (see Kheoh and McLeod 1992; Kwan and Yangru 1996; Brooks 1999; Brooks and Henry 2000;

Chen 2002; Kwan, Sim, and Wu 2002; Kwan 2003; Kwan and Yangru 2003).

Based on the results of the papers mentioned above, I have included some portmanteau test statistics in Tol. There was the original Box–Pierce and the modified Ljung–Box statistics, although they remained hidden. The new statistics available are the Li-McLeod's Q (Section 2.2) Monti's 'Q (Section 2.3), ,McLeod-Li's Q (Section 2.4), and especially, the NDM statistic (Section 2.5). As a byproduct, I have checked the ACF and PACF estimation routines.

2. Portmanteau Statistics

The classical portmanteau test statistic is the one proposed by Box and Pierce

$$Q_{BP} = n \sum_{k=1}^{m} \hat{r}_k^2$$

where $\hat{r}_k$ is the sample autocorrelation of order k of the residual. Under the null hypothesis that the ARMA model is adequate, Q_{BP} is distributed as a χ^2 with $(m-p-q)$ degrees of freedom.

This classical statistic has been widely studied, showing that in finite samples its distribution falls apart from the asymptotic one. This fact has been the starting point of some modifications, being especially important the ones presented in Sections 2.1, and 2.2.

2.1. Ljung-Box

After some discussion about the finite sample distribution of the test statistic proposed by Box and Pierce and its conservative behavior, a modification was proposed. The new test statistic proposed by Ljung and Box is

$$Q_{LB} = n(n+2) \sum_{k=1}^{m} \frac{\hat{r}_k^2}{n-k}. \tag{2.1}$$

The statistic Q_{LB} has a finite sample distribution that is much closer to that of the χ^2_{m-p-q}. The intuition is that we are adjusting each $\hat{r}_k$ in the Q_{BP} statistic by its asymptotic variance. However, this modification is not without criticism, since it has been shown that its variance could be substantially larger than that of its asymptotic distribution.

2.2. Li-McLeod

This is another modification of the Q_{BP} statistic. The Q_{LM} statistic is:

$$Q_{LM} = Q_{BP} + \frac{m(m+1)}{2n} = \frac{m(m+1)}{2n} + n \sum_{k=1}^{m} \hat{r}_k^2 \tag{2.2}$$

One advantage of Q_{LM} is than, unlike Q_{LB} it moves the finite sample distribution of Q_{BP} much closer to its asymptotic distribution without inflating its variance Q_{LM} is also very easy to apply and program although it is less popular that Q_{LB}. It has been demonstrated that Q_{LM} has, in general, a variance that is closer to its asymptotic variance, whereas Q_{LB} is more sensitive with significance levels somewhat larger than the nominal ones when n is large. In contrast, Q_{LM} is slightly conservative. However, the powers of the two tests are almost identical, with the power of Q_{LB} slightly higher. In practice, it is preferable to use the most conservative test when the powers are comparable.

2.3. Monti's Q

Monti (1994) proposed a portmanteau test similar to (2.1) using the residual partial autocorrelations, $\hat{\pi}_k$, $k = 1, 2, \dots, m$. The proposed statistic is

$$Q_M = n(n+2) \sum_{k=1}^{m} \frac{\hat{\pi}_k^2}{n-k}. \tag{2.3}$$

and follows an asymptotic χ^2_{m-p-q} if the fitted $ARMA$ model is adequate. Simulation experiments suggest that the performance of Q_M is comparable to that of Q_{LB} and better if the order of the moving average is understated. On the other hand, Q_{LB} performs better if the order of the autoregressive part is understated.

2.4. McLeod-Li

The idea is to use the squared residual autocorrelation for diagnostic checking for possible departures from the linear ARMA assumption. The lag-k-squared residual autocorrelation is defined by

$$\hat{r}_{aa}(k) = \frac{\sum_{t=k+1}^{n} (\hat{a}_t^2 - \hat{\sigma}^2)(\hat{a}_{t-k}^2 - \hat{\sigma}^2)}{\sum_{t=k+1}^{n} (\hat{a}_t^2 - \hat{\sigma}^2)^2}$$

where $\hat{\sigma} = \sum \hat{a}^2/n$. A goodness-of-fit test is provided by the statistic

$$Q_{ML} = n(n+2) \sum_{i=1}^{m} \frac{\hat{r}_{aa}^2(i)}{n-i} \tag{2.4}$$

It is important to note that under the null hypothesis that the ARMA model alone is adequate, Q_{ML} is distributed asymptotically as a χ^2 with m degrees of freedom

$$\text{Under } H_0 \qquad Q_{ML} \sim \chi^2(m)$$

This is different from the result of previous sections where the test statistics were distributed as $\chi^2(m-p-q)$.

From some simulations studies, it is known that the test provides acceptable results with $m = 20$ for sample sizes as low as 50.

2.5. D_m

Peña and Rodríguez (2002) proposed a portmanteau test by applying a general multivariate measure dependence to the autocorrelation matrix. Let the residual correlation matrix $\hat{R}_m$, of order m, be given by

$$\hat{R}_m = \begin{bmatrix} 1 & \hat{r}_1 & \hat{r}_2 & \dots & \hat{r}_m \\ \hat{r}_1 & 1 & \hat{r}_1 & \dots & \hat{r}_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{r}_m & \hat{r}_{m-1} & \hat{r}_{m-2} & \dots & 1 \end{bmatrix} \tag{2.5}$$

The original test statistic was

$$\hat{D}_m = n\Big(1 - |\hat{R}_m|^{1/m}\Big) \tag{2.6}$$

where it is well known that

$$|\hat{R}_m|^{1/m} = \prod_{i=1}^{n}(1-\hat{\pi}_i^2)^{(m+1-i)/m}$$

The small sample properties are improved if we use the standardized autocorrelation coefficients $\tilde{r}_k$

$$\tilde{r}_k = \sqrt{\frac{n+2}{n-k}}\hat{r}_k \tag{2.7}$$

leading to the modified test statistic

$$D_m = n\Big(1 - |\tilde{R}_m|^{1/m}\Big) \tag{2.8}$$

However, Peña and Rodríguez (2004) propose a modification of the test statistic in order to improve its properties, The new statistic is

$$D_m^\star = -\frac{n}{m+1}\log|\tilde{R}_m|, \tag{2.9}$$

which, taking into account that

$$|\tilde{R}_m| = \prod_{i=1}^{n}(1-\tilde{\pi}_i^2)^{(m+1-i)}$$
$$\tilde{\pi}_i{}^2 = \frac{n+2}{n-i}\hat{\pi}_i{}^2$$

leads to the test statistic

$$D_m^\star = -n\sum_{i=1}^{m}\frac{m+1-i}{m+1}\log\Big(1-\frac{n+2}{n-i}\,\hat{\pi}^2\Big) \tag{2.11}$$

Asymptotic distribution

There are two different approaches to the asymptotic distribution of the test statistic (2.11). The first one was proposed by Peña and Rodríguez (2002) and is based on the Gamma distribution. The test statistic follows asymptotically a Gamma distribution with parameters α and β, where

$$\alpha = \frac{3(m+1)[m-2(p+q)]^2}{2[2m(2m+1)-12(m+1)(p+q)]} \tag{2.12a}$$

$$\beta = \frac{3(m+1)[m-2(p+q)]}{2m(2m+1)-12(m+1)(p+q)} \tag{2.12b}$$

and the distribution has mean $\alpha/\beta = m/2-(p+q)$ and variance $\alpha/\beta^2 = m(2m+1)/(3(m+1))-2(p+q)$. We denote $GD_m^\star$ this first approximation which is distributed as a $\mathcal{G}(\alpha,\beta)$.

Note 1. This is important. The implementation in Octave (and possibly in Matlab) is different to that in R or using the GSL libraries, where the second parameter should be in fact $1/\beta$. I must still make sure of this point.

In Octave the pdf of a Gamma distribution of parameters (a, b) is

$$f(x) = b^a \, x^{a-1} \, \exp(-bx)/\Gamma(a),$$

whereas in R and GSL the pdf of a Gamma distribution of parameters (a, s) is

$$f(x) = \frac{1}{\Gamma(a)s^a} x^{a-1} \exp(-x/s).$$

Given that

$$\mathsf{E}(X) = as \qquad \mathsf{VAR}(X) = as^2$$

It must be

$$a = \alpha \qquad s = 1/\beta.$$

The second approach, proposed by Peña and Rodríguez (2004) is based on the Normal distribution. The test statistic is

$$ND^\star_m = (\alpha/\beta)^{-1/\lambda}(\lambda/\sqrt{\alpha})\left\{\left(D^\star_m\right)^{1/\lambda} - (\alpha/\beta)^{1/\lambda}\left[1 - \frac{1}{2\alpha}\left(\frac{\lambda-1}{\lambda^2}\right)\right]\right\} \tag{2.13}$$

where

$$\lambda = \left\{\frac{2\Big[m/2 - (p+q)\Big]\Big[m^2/\Big(4(m+1)\Big) - (p+q)\Big]}{3\Big[m(2m+1)/\Big(6(m+1)\Big) - (p+q)^2\Big]}\right\}^{-1}$$

and α and β are given by equations (2.12a)–(2.12b). However, it is found that for moderate large values of m we get $\lambda \approxeq 4$. The statistic $ND^\star_m$ from (2.13) with $\lambda = 4$ is the second approximation which is distributed as a $\mathsf{N}(0, 1)$.

Checking the linearity assumption

Analogously to Section 2.4, where a portmanteau test is used to detect nonlinearity when applied to the squared residuals of the model, Peña and Rodríguez (2002); Peña and Rodríguez (2004) generalize their test for testing for nonlinearity. Their test could be applied to the squared residuals, the absolute values of the residuals and the log of one minus the squared residuals, although it seems that when applied to the absolute value of the residuals is more powerful in finding nonlinearity in heteroscedastic models (such as GARCH and Stochastic Volatility models). The test applied to the absolute value of the residuals is

$$\hat{D}^\star_m(|\hat{\epsilon}|) = -\frac{n}{m+1}\log|\hat{R}_m(|\hat{\epsilon}|)|,$$

where $\hat{R}_m(|\hat{\epsilon}|)$ is the autocorrelation matrix (2.5) which is now built using the autocorrelation coefficients of the absolute residuals, $\tilde{r}_k(|\hat{\epsilon}|)$, given by

$$\tilde{r}_k(|\hat{\epsilon}|) = \frac{n+2}{n-k}\frac{\sum_{i=k+1}^n(|\hat{\epsilon}_i| - |\hat{\sigma}|)(|\hat{\epsilon}_{i+k}| - |\hat{\sigma}|)}{\sum_{i=1}^n(|\hat{\epsilon}_i| - |\hat{\sigma}|)},$$

being $|\hat{\sigma}| = \sum|\hat{\epsilon}|/n$. The asymptotic distribution of $\hat{D}^\star_m(|\hat{\epsilon}|)$ is similar to that of $D^\star_m$ as explained in Section 2.5.1. Notice, however, that now we make $p = q = 0$ when obtaining the corresponding values of α, β, and λ.

Note 2. Notice that this is exactly what happened with the Q_{ML} test statistic, where the number of parameters was not deducted from the degrees of freedom of the asymptotic χ^2 distribution (see (2.4) in Section 2.4).

A. PACF estimation

The eestimation of the PACF of up to order k involves 2 steps:

i) The estimation of the ACF up to order k

ii) The solution of the Yule–Walker equations of up to order k

The ACF function has been modified so that it takes deviations from the mean by default. This was a problem y autocovariance function estimation functions, which by default did not demean the series.

In order to estimate the PACF of up to order k we must solve systems of equations. For $j = 1$ we know that tha ACF and the PACF must be equal. For $j = 1, 2 \ldots, k$ we have the following system of equations:

$$\begin{array}{c} \rho_1 = \phi_{k1}\rho_0 + \phi_{k2}\rho_1 + \ldots + \phi_{kk}\rho_{k-1} \\ \rho_2 = \phi_{k1}\rho_1 + \phi_{k2}\rho_0 + \ldots + \phi_{kk}\rho_{k-2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \rho_k = \phi_{k1}\rho_{k-1} + \phi_{k2}\rho_{k-2} + \ldots + \phi_{kk}\rho_0 \end{array} \tag{A.1}$$

- $k = 1$ $\quad \rho_1 = \phi_{11}\rho_0 \quad \Longrightarrow \phi_{11} = \rho_1$
- $k = 2$

$$\begin{array}{c} \rho_1 = \phi_{21}\rho_0 + \phi_{22}\rho_1 \\ \rho_2 = \phi_{21}\rho_1 + \phi_{22}\rho_0 \end{array} \quad \Longrightarrow \phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}}$$

- $k = 3$

$$\begin{array}{c} \rho_1 = \phi_{31}\rho_0 + \phi_{32}\rho_1 + \phi_{33}\rho_2 \\ \rho_2 = \phi_{31}\rho_1 + \phi_{32}\rho_0 + \phi_{33}\rho_1 \\ \rho_3 = \phi_{31}\rho_2 + \phi_{32}\rho_1 + \phi_{33}\rho_0 \end{array} \quad \Longrightarrow \phi_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}}$$

Hence the procedure is the following:

i) Calculate the ACF up to order k

ii) Build the system of equations (A.1)

iii) For $j = 2, \ldots, k$ solve the subsystems os equations obtained form (A.1). The solution is implemented using the QR method.

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